Orbit determination from two position vectors by the continuation-method optimal parametrization
by the continuation method optimal parametrization

V. Kuznetcov[1]

[1]Institute of Applied Astronomy, Russian Academy of Sciences, Saint Petersburg

The old classic problem of finding an orbit of a celestial body from two position vectors at two instants of time is considered. The history of the problem is more than two centuries old [1] and there are many approaches for finding the solution. The current investigation i s based on the Shefer method [2], giving a solution that is free from uncertainties and may be applied to general Keplerian motion. This method uses a single equation with one unknown x. It may be written, in a common case, as:

$$\sqrt{r_1 + 2\sqrt{r_1 r_2}\cos\frac{\theta_{21}}{2}(2x-1)+r_2}\left(\sqrt{2r_1 r_2}\cos\frac{\theta_{21}}{2}+\frac{\sqrt{2}}{3}\left(r_1+2\sqrt{r_1 r_2}\cos\frac{\theta_{21}}{2}(2x-1)+r_2\right)X(x)\right)=\tau_{21}, \quad (1)$$

where θ_{21} is the angle between the vectors $\mathbf{r}_1$ and $\mathbf{r}_2$, $\tau_{21} = k(t_2 - t_1)$, k is the Gauss constant or its analog for another center of attraction, and $X(x) = F(1, 3, 5/2; x)$ is the hypergeometric function. For solving equation (1), the application of the Newton-Raphson method with a choice for an initial approximate solution was suggested [2]. Alternatively, the use of the continuation method with the best parametrization is proposed [3,4]. For this, on the basis of the global homotopy, we will develop an analog of equation (1). It depends on the parameter of homotopy $\mu \in [0,\ 1]$ and the initial values of the problem. The solution of such extended equation is produced through the generation of a system of ordinary differential equations with initial conditions. This method was suggested by Davidenko [5] and it is named as continuous continuation. The optimal parameter of continuation is s — the length of arc along the current solution curve. As an initial value for x, we shall take $x = 0$ — the value for a parabolic orbit. Thus, the problem is reduced to solving the following Cauchy problem:

$$\left.\begin{aligned}
&\tfrac{dx}{ds} = -\tfrac{\sqrt{2}}{3}\sqrt{r_1 - 2\sqrt{r_1 r_2}\cos\tfrac{\theta_{21}}{2} + r_2}\left(r_1 + \sqrt{r_1 r_2}\cos\tfrac{\theta_{21}}{2} + r_2\right) + \tau_{21},\\
&\tfrac{d\mu}{ds} = -2^{3/2}\Bigg[\tfrac{Z(x)}{5}\left(r_1 + 2\sqrt{r_1 r_2}\cos\tfrac{\theta_{21}}{2}(2x-1) + r_2\right)^{3/2} +\\
&+\sqrt{r_1 r_2}\cos\tfrac{\theta_{21}}{2}X(x)\left(r_1 + 2\sqrt{r_1 r_2}\cos\tfrac{\theta_{21}}{2}(2x-1) + r_2\right)^{1/2} +\\
&+r_1 r_2\cos^2\tfrac{\theta_{21}}{2}\left(r_1 + 2\sqrt{r_1 r_2}\cos\tfrac{\theta_{21}}{2}(2x-1) + r_2\right)^{-1/2}\Bigg]
\end{aligned}\right\}, \quad (2)$$

where $Z(x) = F(2, 4, 7/2; x)$ is the hypergeometric function, with initial conditions at the point $s = 0$: $x(0) = 0$, $\mu(0) = 1$. It will be necessary to integrate (2) in the direction of increasing parameter s until $\mu = 0$ is obtained. The corresponding x is the desired solution. This algorithm loses efficiency and reliability at $x \to 1$ or $\theta_{21} \to 2\pi$.

References: [1] Gauss C. F. Theoria motus corporum coelestium in sectionibus conics solem ambientium. Hamburg: Perthes und Besser, 1809, 227 p. [2] Shefer V. A. New method of orbit determination from two position vectors based on solving Gauss's equations. Astronomicheskii Vestnik, 2010, Vol. 44, no 3, pp. 273–288. [3] Shalashilin V. I., Kuznetsov E. B. Parametric Continuation and Optimal Parametrization in Applied Mathematics and Mechanics. Kluwer, 2003. [4] Kuznetsov E. B. Best parameterization in curve construction. Zh. Vychisl. Mat. Mat. Fiz., 2004, Vol. 44, no 9, pp. 1540–1551. [5] Davidenko D. F. On a new method of numerical solution of systems of nonlinear equations. Dokl. Akad. Nauk. SSSR. Vol. 88, no 4, 1953, pp. 601–602.