Modelling polarization dependent absorption: The vectorial Lambert-Beer law

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The scalar Lambert-Beer law, describing the absorption of unpolarized light travelling through a linear non-scattering medium, is simple, well-known, and mathematically trivial. However, when we take the polarization of light into account and consider a medium with polarization dependent absorption, we now need a Vectorial Lambert-Beer Law (VLBL) to quantify this interaction. Such a generalization of the scalar Lambert-Beer law appears not to be readily available.

A careful study of this topic reveals that it is not a trivial problem. We will see that the VLBL is not and cannot be a straightforward vectorized version of its scalar counterpart. The aim of the work is to present the general form of the VLBL and to explain how it arises.

A reasonable starting point to derive the VLBL is the Vectorial Radiative Transfer Equation (VRTE), which models the absorption and scattering of (partially) polarized light travelling through a linear medium. When we turn off scattering, the VRTE becomes an infinitesimal model for the VLBL holding in the medium. By integrating this equation, we expect to find the VLBL. Surprisingly, this is not the end of the story. It turns out that light propagation through a medium with polarization-dependent absorption is mathematically not that trivial.

The trickiness behind the VLBL can be understood in the following terms. The matrix in the VLBL, relating any input Stokes vector to the corresponding output Stokes vector, must necessarily be a Mueller matrix. The subset of invertible Mueller matrices forms a Lie group. It is known that this Lie group contains the ortho-chronous Lorentz group as a subgroup. The group manifold of this subgroup has a (well-known) non-trivial topology. Consequently, the manifold of the Lie group of Mueller matrices also has (at least the same, but likely a more general) non-trivial topology (the full extent of which is not yet known). The type of non-trivial topology, possessed by the manifold of (invertible) Mueller matrices and which stems from the ortho-chronous Lorentz group, already implies (by a theorem from Lie group theory) that the infinitesimal VRTE model for the VLBL is not guaranteed to produce in general the correct finite model (i.e., the VLBL itself) upon integration. What happens is that the non-trivial topology acts as an obstruction that prevents the (matrix) exponential function to reach the correct Mueller matrix (for the medium at hand), because it is too far away from the identity matrix. This means that, for certain media, the VLBL obtained by integrating the VRTE may be different from the VLBL that one would actually measure. Basically, we have here an example of a physical problem that cannot be completely described by a differential equation!

The following more concrete example further illustrates the problem. Imagine a slab of matter, showing polarization dependent absorption but negligible scattering, and consider its Mueller matrix for forward propagating plane waves. Will the measured Mueller matrix of such a slab always have positive determinant? There is no apparent mathematical nor physical reason why this (or any) Mueller matrix must have positive determinant. On the other hand, our VRTE model with scattering turned off will always generate a Mueller matrix with positive determinant. This particular example also presents a nice challenge and opportunity for the experimenter: demonstrate the existence of a medium of the envisioned type having a Mueller matrix with non-positive determinant!

Lie group theory not only explains when and why we cannot trust a differential equation, but also offers a way out of such a situation if it arises. Applied to our problem, Lie group theory in addition yields the general form of the VLBL. More details will be given in the presentation.