# Electromagnetic wave scattering from particles of arbitrary shapes using the *Sh*-matrix technique

D. Petrov[*,1], Yu. Shkuratov[1], and G. Videen[2]

[1] *Astronomical Institute of Kharkov V.N. Karazin National University, 35 Sumskaya St, Kharkov, 61022, Ukraine.*
[2] *Army Research Laboratory AMSRD-ARL-CI-EM, 2800 Powder Mill Road Adelphi Maryland 20783 USA.*

A general analytical solution of the light scattered from arbitrarily shaped particles that are represented by an expansion of a series of trigonometric functions and associated Legendre polynomials is considered. We focus here on the Gaussian random particles calculating the *Sh*-matrix elements; the *Sh*-matrix approach is a variety of the *T*-matrix formulation.

## INTRODUCTION

The most fundamental characteristic that determines scattering of electromagnetic waves from physical objects is particle shape. Within the last few decades, there have been many studies of light scattering of electromagnetic waves as a function of particle shape, and significant progress has been achieved in the development of different algorithms and techniques in electromagnetic wave scattering [e.g., 1]. Our approach allows a simplification and unification of the morphological description of particles. We combine this particle description with a *T*-matrix formulation to calculate the light-scattering from such particles using the *Sh*-matrix that can often provide an analytical solution. The *Sh*-matrix technique [2] has been applied to study light scattering of particles with different shapes. The simplified *Sh* matrix depends only on particle morphology and is found by performing surface integrals. Size and refractive index dependence are incorporated through analytical operations on the *Sh* matrix to produce the *T* matrix. We here present the particle generation technique and provide several examples of light scattering calculations using the *Sh*-matrix method. We focus on Gaussian-random-sphere particles.

## THEORY

Let us consider a particle whose shape is described by a single-valued continuous function $R(\theta,\varphi)$, where $\theta$ and $\varphi$ are the polar and azimuth angles, respectively, in a spherical coordinate system with the center located within the particle. In our approach we expand $R(\theta,\varphi)$ in a Laplace series

$$R(\theta,\varphi)=\sum_{l=0}^{\infty}\sum_{m=0}^{l}P_l^m(\cos\theta)(a_{lm}\cos m\varphi+b_{lm}\sin m\varphi), \quad (1)$$

[*] Corresponding author: Dmitry Petrov (petrov@astron.kharkov.ua)

where the expansion coefficients $a_{lm}$ and $b_{lm}$ determine the particle shape. The spherical harmonics $P_l^m(\cos\theta)\cos m\varphi$ and $P_l^m(\cos\theta)\sin m\varphi$ are a complete set of orthonormal functions, and hence the set forms an orthonormal basis of the Hilbert space of square-integrable functions. On the unit sphere, any square-integrable function can be expanded as a linear combination of the functions.

To apply the expansion in practice, the upper limit of the outer summation in (1) should be truncated to finite *N*. The coefficients $a_{lm}$ and $b_{lm}$ can be found, analytically, if the function $R(\theta,\varphi)$ is known. However, there is another way to find the coefficients using discrete values of the function $R(\theta_i,\varphi_j)$. Let us consider the values at $2N^2$ points: $R_{ij} = R(\theta_i,\varphi_j)$, $i = 1\ldots N;\ j = 1\ldots 2N$. By substituting $R(\theta_i,\varphi_j)$ into Eq. (1) with finite upper limit we obtain a system of linear equations, where $a_{lm}$ and $b_{lm}$ are unknown. Solution of this system gives us a representation of the particle shape as a simple and easily calculated expansion into a series expansion over trigonometric functions and associated Legendre's polynomials. Such an approach does not require an explicit form of the function $R(\theta,\varphi)$: it is sufficient to designate its values at $2N^2$ points. For example, one can generate numerically a group of radii radiating more-or-less isotropically from the center of the spherical coordinate system. The radii can be considered as a skeleton of a model particle. The length distribution of the radii and orientation of each are arbitrary depending on the particle. Thus, our approach can include regular particles (e.g., spheroids, ellipsoids and cubes) as well as particles with random shapes, like the Gaussian random particles [3]. To find the light-scattering solution, we introduce the so-called *Sh* matrix, which depends on the object shape only. The elements of the *T* matrix [1] can be expressed in terms of *Sh*-matrix elements. For example, the elements $RgJ^{11}_{mnm'n'}$ and $J^{11}_{mnm'n'}$ can be expressed using the *Sh* matrix as follows [2]:

$$RgJ^{11}_{mnm'n'}(X,m_0) = X^{n+n'+2}(m_0)^{n'} \sum_{k_1=0}^{\infty} \frac{(X\cdot m_0)^{2k_1}}{k_1!\,\Gamma\left(n'+k_1+\frac{3}{2}\right)} \sum_{k_2=0}^{\infty} \frac{(X)^{2k_2}}{k_2!\,\Gamma\left(n+k_2+\frac{3}{2}\right)} \cdot RgSh^{11}_{mnm'n',k_1+k_2}, \quad (2)$$

$$J^{11}_{mnm'n'}(X,m_0) = X^{-n+n'+1}(m_0)^{n'} \sum_{k_1=0}^{\infty} \frac{(X\cdot m_0)^{2k_1}}{k_1!\,\Gamma\left(n'+k_1+\frac{3}{2}\right)} \sum_{k_2=0}^{\infty} \frac{(X)^{2k_2}}{k_2!\,\Gamma\left(-n+k_2+\frac{1}{2}\right)} \cdot Sh^{11}_{mnm'n',k_1+k_2}, \quad (3)$$

where $X = 2\pi r/\lambda$ is the size parameter, *r* is the size of the major axis of a particle, $\lambda$ is the wavelength of incident light; $m_0$ is the refractive index of the particle, *Sh* and *RgSh* are the shape matrices or just *Sh*-matrix elements. We have found explicitly the elements for any particle whose shape can be represented by the expansion (1). Thus we suggest an analytical solution for a very wide class of particle shapes.

## CALCULATIONS AND DISCUSSION

The current implementation of the algorithm for computing the expansion coefficients and particle scattering properties is written in the C++ language and for proper operation requires the Intel Math Kernel Library not older than 9.1. This code allows us to calculate the expansion coefficients for different particles, e.g., ellipsoids, parallelepiped-like particles, and Gaussian random spheres. The latter particles can be described through the spherical harmonics and the associated Legendre polynomials in the manner [3]

$$R(\theta,\varphi)=\frac{C\exp\left(\sum_{l=0}^{N}\sum_{m=0}^{l}P_l^m(\cos\theta)(A_{lm}\cos m\varphi+B_{lm}\sin m\varphi)\right)}{\sqrt{1+\sigma^2}}, \quad (5)$$

where the coefficients $A_{lm}$ and $B_{lm}$ are independent Gaussian random variables with zero mean and equal variances:

$$\beta_{lm}^2=(2-\delta_{m0})\frac{(l-m)!}{(l+m)!}c_l\beta^2, \quad (6)$$

$$\beta^2=\ln(1+\sigma^2), \quad (7)$$

$$c_l=(2l+1)\exp(-\kappa)i_l(\kappa), \quad (8)$$

$$\kappa=\frac{1}{4}\left(\sin\frac{\Gamma}{2}\right)^{-2}, \quad (9)$$

where $C$ is a constant, $\sigma^2$ is the radii variance, $\Gamma$ is the correlation angle, and $i_l(\kappa)$ are the modified spherical Bessel functions. This method allows one to generate irregular particles. The value $\Gamma$ should be entered in degrees and cannot be smaller than 3°.

In Figs. 1 and 2, we show orientation-averaged calculated phase dependences of intensity and polarization degree for Gaussian random spheres at different $m_0$ and $\Gamma$ using the *Sh*-matrix method.

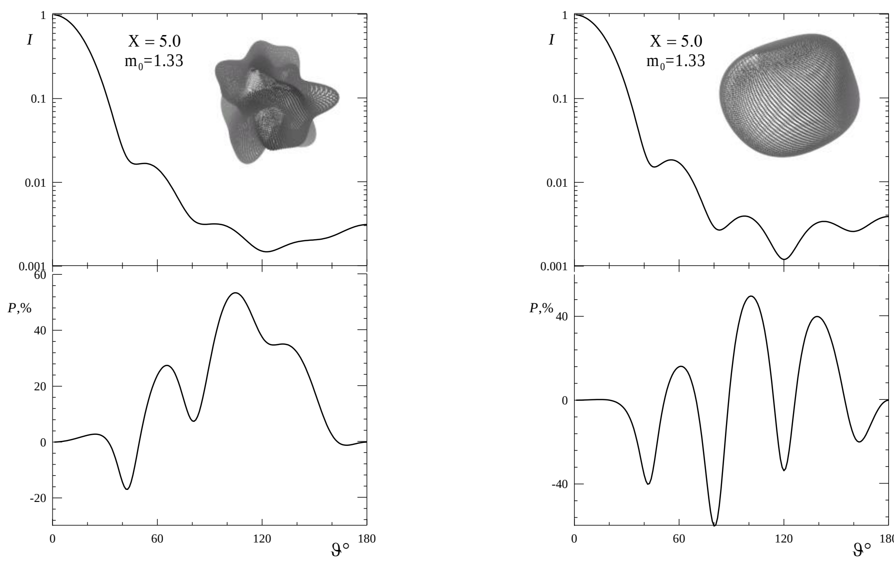


**Figure 1.** Phase curves of intensity and polarization degree for a Gaussian random sphere having $X = 5$, $m_0 = 1.33$, $\sigma = 0.3$, $\Gamma = 10°$ (left panel), and $\Gamma = 50°$ (right panel).

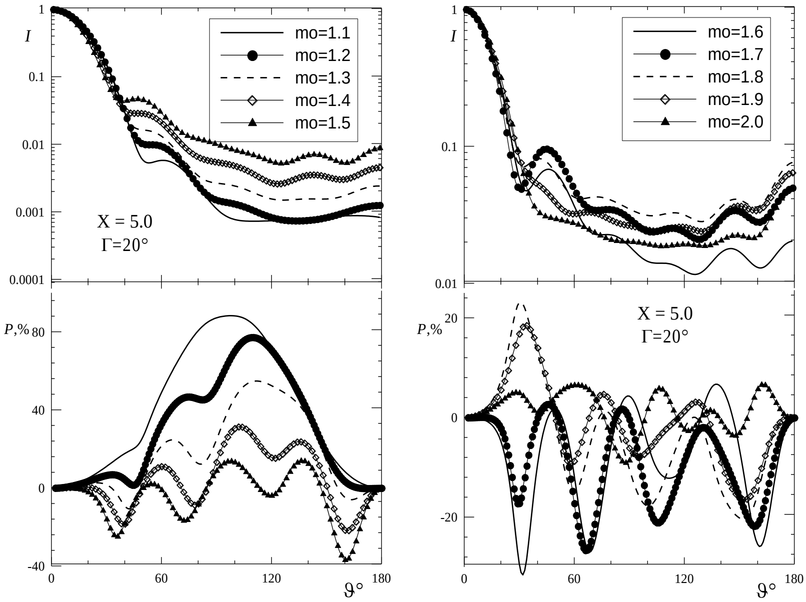


**Figure 2.** Phase curves of intensity and polarization degree for a Gaussian random sphere having $\sigma = 0.3$ and $\Gamma = 20°$ at different $m_0$.

**Acknowledgments:** This work was supported by the Army Medical Research Institute of Chemical Defense (Lucille Lumley, PhD) under the auspices of the U.S. Army Research Office Scientific Services Program administered by Battelle (Delivery Order 0378, Contract No. W911NF-07-D-0001).

## REFERENCES

[1] M.I. Mishchenko, L.D. Travis, and A.A.Lacis. *Scattering, absorption, and emission of light by small particles*. Cambridge: Cambridge University Press (2002).

[2] D. Petrov, E. Synelnyk, Y. Shkuratov, and G. Videen. The *T*-matrix technique for calculations of scattering properties of ensembles of randomly oriented particles with different size. JQSRT **102** (2006).

[3] K. Muinonen. Light scattering by Gaussian random particles: Rayleigh and Rayleigh-Gans approximations JQSRT **55** (1996).