# The need for a first-order quasi-Lorentz transformation

D. Censor*

*Dept. of Electrical and Comp. Engineering, Ben-Gurion University, Beer Sheva, 84105, Israel.*

Solving electromagnetic scattering problems involving non-uniformly moving objects requires an approximate but consistent extension of Einstein's Special Relativity theory which originally is valid for constant velocities only. For moderately varying velocities, a quasi-Lorentz transformation is presented. The conditions for form-invariance of the Maxwell equations, the so-called "principle of relativity", are shown to hold for a broad class of motional modes and time scales. An example of scattering by a harmonically oscillating mirror is given. The present extended abstract is an overview of the full draft [1], where the derivations and formulas are given explicitly.

## INTRODUCTION AND RATIONALE

Einstein's Special Relativity (SR) theory [2] is based on the invariance of $c$, the speed of light in free space (vacuum), for all inertial observers. The other postulate is the so-called "principle of relativity" (PR) for the Maxwell equations (ME). From the invariance of $c$ follow [2] the spatial and temporal Lorentz Transformations (LT) (Eq. 1 in [1]), involving inertial reference systems $\mathbf{\Gamma}$ and $\mathbf{\Gamma}'$ and their associated space-time coordinates $\mathbf{r}, t$, and $\mathbf{r}', t'$, respectively.

Henceforth, inverse formulas will be referred to by an asterisk on the initial equation number, e.g., Eq. 1* (in [1]). Typically, inverse transformations are derived by solving the initial formulas for the sought variables. Formally, they follow by interchanging primed and unprimed symbols where $\mathbf{v}' = -\mathbf{v}$.

So far the LT (Eq. 1 in [1]) is restricted to constant velocities $\mathbf{v}$. If we try to formulate a systematic model for varying velocities, at least one reference frame ceases to be inertial. We cannot claim that such an approximate model is applicable to arbitrarily large velocities. Therefore, for varying velocities, the model will be restricted to a first order approximation in the normalized velocity $\beta = v/c$. Furthermore, for the model to be simple and tractable, the effects of the varying velocities are viewed as kinematical only, i.e., no attempt will be made to incorporate acceleration into media properties and/or the ME. For sufficiently small accelerations, this seems to provide a reasonable theoretical framework.

A very suggestive methodology for generalizing the problem to varying velocities is to approximate an inhomogeneously moving reference frame as a sequence of systems moving with different local/instantaneous velocities $\bar{\mathbf{v}} = \bar{\mathbf{v}}(\bar{\mathbf{r}}, \bar{t})$, instead of the continuously changing $\mathbf{v} = \mathbf{v}(\mathbf{r}, t)$. The velocity $\bar{\mathbf{v}}$ is assumed to be valid for a limited span of space-time in the vicinity of the world event $\bar{\mathbf{r}}, \bar{t}$. When the discrepancy becomes too large, the

* Corresponding author: Daniel Censor (censor@ee.bgu.ac.il)

velocity is updated. Thus, the continuous change of $\mathbf{v}=\mathbf{v}(\mathbf{r},t)$ is replaced by a sequence of discrete constant velocities.

As a multi-scale method, this is plausible for systems, where the velocity changes slowly and monotonically over space and time regions large in comparison to other system parameters. However, this assumption holds only for a restricted class of configurations. It does not hold, for example, in the case of a wave emitted by oscillating antennas or scatterers. Intuitively one expects the antenna motion to cause a Doppler effect which during the mechanical cycle creates higher and lower frequencies. The results would then be akin to a frequency modulated carrier wave of frequency $\omega$, creating sidebands of frequencies $\omega \pm n\Omega$, with $\Omega$ corresponding to the mechanical frequency. This is a result that cannot be displayed when the local instantaneous $\bar{\mathbf{v}}$ is employed, because for each incremental change of $\bar{\mathbf{v}}$ a different Doppler frequency shift is created, and the resulting continuous spectrum does not merge into the expected discrete sideband frequencies.

Our goal is to find a first-order quasi-Lorentz transformation (QLT) that will satisfactorily deal with scattering by objects moving at varying velocities.

## FIRST-ORDER QUASI-LORENTZ TRANSFORMATION

The first-order SR is derived by keeping in the LT and ME only the first-order terms in $\beta$. For varying velocities we use a space-time dependent velocity $\mathbf{v}(\mathbf{r},t)$. Thus, the differential first-order QLT is *postulated* as

$$d\mathbf{r}' = d\mathbf{r} - \mathbf{v}(\mathbf{r},t)dt, dt' = dt - \mathbf{v}(\mathbf{r},t)\cdot d\mathbf{r}/c^2 \qquad \text{(8a-b)}$$

The corresponding integral forms are only valid when it is possible to return to the initial differential formula (Eq. 8). This prescribes that the line integrals be determined by the integration end points. Otherwise, the Leibniz rule for differentiation of integrals is not applicable. Hence the velocity field must be conservative.

## THE PRINCIPLE OF RELATIVITY

Einstein [2] postulated PR, namely the form-invariance of ME. Thus, in $\mathbf{\Gamma}$ the ME are given by

$$\partial_{\mathbf{r}} \times \mathbf{E} = -\partial_t \mathbf{B}, \partial_{\mathbf{r}} \times \mathbf{H} = \partial_t \mathbf{D}, \partial_{\mathbf{r}} \cdot \mathbf{D} = 0, \partial_{\mathbf{r}} \cdot \mathbf{B} = 0 \qquad \text{(12a-d)}$$

with fields that are space and time dependent, e.g., $\mathbf{E}=\mathbf{E}(\mathbf{r},t)$. According to PR, in another inertial system $\mathbf{\Gamma}'$, the form-invariance of ME (Eq. 12) is preserved. Thus, the fields are $\mathbf{E}'=\mathbf{E}'(\mathbf{r}',t')$ etc. in terms of the appropriate native coordinates in $\mathbf{\Gamma}'$.

The PR *statement* associating Eq. 12, Eq. 12*, is incomplete unless the LT relating the fields is provided, facilitating the derivation of the Field Transformation (FT) formulas. Here we need the FT only to order $\beta$. The chain rule of calculus is applied, specifying an LT for the differential operators (Eq. 14 in [1]). Substituting Eq. 14 (in [1]) in the inverse ME (Eq. 12*) and collecting terms, Eq. 12 is obtained subject to the FT to the first order in $\beta$,

$$\mathbf{E}' = \mathbf{E} + \mathbf{v} \times \mathbf{B}, \mathbf{B}' = \mathbf{B} - \mathbf{v} \times \mathbf{E} / c^2, \mathbf{D}' = \mathbf{D} + \mathbf{v} \times \mathbf{H} / c^2, \mathbf{H}' = \mathbf{H} - \mathbf{v} \times \mathbf{D}. \quad \text{(20a-d)}$$

## MAXWELL-EQUATIONS FORM INVARIANCE FOR VARYING VELOCITIES

To the first order in $\beta$, application of the chain rule for varying velocity $\mathbf{v} = \mathbf{v}(\mathbf{r}, t)$ to Eq. 12d* yields Eq. 12d, and similarly for Eq. 12c* and Eq. 12c, respectively. For the vector equations 12a*, 12a, and 12b*, 12b, this holds only if the wave angular frequency is much smaller than the typical velocity variation

$$\partial_t \mathbf{v} \ll \omega \mathbf{v} \quad (25)$$

In cases where Eq. 25 holds, the FT in Eq. 20 apply by replacing the constant $\mathbf{v}$ with $\mathbf{v} = \mathbf{v}(\mathbf{r}, t)$.

## PLANE-WAVE TRANSFORMATIONS

Plane waves are characterized by constant vector amplitudes, with the space and time variation delegated to appropriate scalar phase exponential $e^{i\theta}$. Hence the FT (Eq. 20) are applied to the constant vector amplitudes, and the phase is a relativistic scalar invariant

$$\theta = \mathbf{k} \cdot \mathbf{r} - \omega t = \theta' = \mathbf{k}' \cdot \mathbf{r}' - \omega' t' \quad (28)$$

leading to the SR Fresnel drag and Doppler shift formulas, respectively. To first order in $\beta$

$$\mathbf{k}' = \mathbf{k} - \mathbf{v}\omega / c^2, \omega' = \omega - \mathbf{v} \cdot \mathbf{k} \quad \text{(30a-b)}$$

Replacing Eq. 28 by

$$d\theta = \mathbf{k} \cdot d\mathbf{r} - \omega dt = d\theta' = \mathbf{k}' \cdot d\mathbf{r}' - \omega' dt' \quad (31)$$

and substituting the QLT Eq. 8 yields Eq. 30 with space-time dependent parameters.

## SCATTERING BY A PLANE OSCILLATING MIRROR

A simple example of a harmonic plane wave normally incident on a plane oscillating mirror is analyzed. Free space (vacuum), normal incidence, and a perfect mirror are assumed. The mirror is at rest in $\Gamma'$ at $z' = 0$, moving harmonically with respect to the origin of $\Gamma$.

The incident wave is given in $\mathbf{\Gamma}$ by

$$\mathbf{E} = \hat{\mathbf{x}}E_0 e^{i\theta}, \mathbf{H} = \hat{\mathbf{y}}H_0 e^{i\theta}, E_0 / H_0 = (\mu_0 / \varepsilon_0)^{1/2} = Z, \theta = kz - \omega t . \tag{37}$$

In $\mathbf{\Gamma}'$, where in terms of order $\beta_0$ we may use either $t$ or $t'$, the phase is given by

$$\begin{aligned} &\theta' = \int d\theta' = \int_{z_0'}^{z'} k'(t')d\underset{\sim}{z}' - \int_{t_0'}^{t'} \omega'(\underset{\sim}{t}')d\underset{\sim}{t}' = k'z' - \omega t' + (\omega\beta_0 / \Omega)\sin\Omega t' \\ &k' = k - (\omega v_0 / c^2)\cos\Omega t = kP, P = 1 - \beta_0 \cos\Omega t = 1 - \beta_0 \cos\Omega t' \end{aligned} \tag{38a-b}$$

The amplitudes are transformed according to

$$\begin{aligned} &\mathbf{E}' = \hat{\mathbf{x}}E_0' e^{i\theta'} = E_0 \hat{\mathbf{x}}P e^{i\theta'}, \mathbf{H}' = \hat{\mathbf{y}}H_0' e^{i\theta'} = H_0 \hat{\mathbf{y}}P e^{i\theta'} \\ &E_0' / E_0 = H_0' / H_0 = P, E_0' / H_0' = Z \end{aligned} \tag{39}$$

At the mirror location $z' = 0$, the fields are time-varying in accordance with

$$M = Pe^{-i\omega t' + i\xi_0 \sin\Omega t'}, \xi_0 = \omega\zeta_0 / c = \omega\beta_0 / \Omega , \tag{40a}$$

vindicating the assertion above that, at the oscillating mirror, we have a new spectrum.

In order to calculate the reflected wave and transform it back from $\mathbf{\Gamma}'$ to $\mathbf{\Gamma}$, we first need to represent the spectrum as a superposition of time-harmonic signals. Expanding the exponential in Eq. 40a in a series involving Bessel functions, we get

$$\begin{aligned} &M = Pe^{-i\omega t'}\Sigma_n J_n e^{in\Omega t'} = P\Sigma_n J_n e^{-i\omega_n' t'} \\ &\omega_n' = \omega - n\Omega, \Sigma_n = \Sigma_{n=-\infty}^{n=\infty}, J_n = J_n(\xi_0) \end{aligned} \tag{40b}$$

Finally, the reflected wave, constituting a superposition of harmonic plane waves, is

$$\begin{aligned} &\mathbf{E}' = \hat{\mathbf{x}}E_0 \Sigma_n I_n e^{i\theta_n'}, \mathbf{H}' = \hat{\mathbf{y}}H_0 \Sigma_n I_n e^{i\theta_n'}, \theta_n' = k_n' z' - \omega_n' t' \\ &I_n = (J_n - \beta_0 (J_{n-1} + J_{n+1}) / 2) = (1 - \beta_0 n / \xi_0) J_n = (1 - n\Omega / \omega) J_n \end{aligned} \tag{39b}$$

## REFERENCES

[1] http://www.ee.bgu.ac.il/~censor/quasi-lorentz.pdf

[2] A. Einstein, "Zur Elektrodynamik bewegter Körper", *Ann. Phys. (Lpz.)*, Vol. 17, 891-921, 1905; English translation: "On the Electrodynamics of moving bodies", *The Principle of Relativity*, Dover.