Electromagnetic scattering by an inhomogeneous circular cylinder using fast convergent series expansions

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In this paper we examine the scattering of a TM plane wave by an infinite circular cylinder having inhomogeneous optical properties e.g. ρ–varying permittivity $\epsilon(\rho)$. The method is based on constructing the volume integral equation and then expanding the unknown functions in Dini's series, which have the characteristic of being fast convergent. Numerical results are given for the various values of the parameters.

INTRODUCTION

Electromagnetic scattering problems of main interest are those which present structures having irregular optical properties. In [1], a circular cylinder with inhomogeneous cladding is examined using electric and magnetic current distributions while in [2], scattering from inhomogeneous bodies using a new boundary method is presented.

In the present work we study the scattering of a TM plane wave by a circular infinite dielectric cylinder of radius a with varying permittivity $\epsilon(\rho)$. The geometry of the scatterer is shown in Fig. 1.

In nonhomogeneous media, Helmholtz equation has the extra term [3] $\nabla[2\,\nabla n(\vec{r})\cdot \vec{E}(\vec{r})/n(\vec{r})]$, with $n(\vec{r})$ being the refraction index of the medium. By selecting a slow varying profile for $n(\vec{r})$, the term can be omitted and therefore, this concession leads to the well known homogeneous Helmholtz equation. Then we try to solve it by constructing the volume integral equation.

E-WAVE POLARIZATION

Formulation of the problem

By applying the surface equivalence theorem [4] to the configuration of Fig. 1 and then, by making use of the reaction theorem [4], one can arrive at the desired integral equation

$$E_z(\vec{\rho}) = E_z^{\mathrm{inc}}(\vec{\rho}) + j\frac{k_1^2}{\omega\mu_1}\iint_S \left[\frac{k_2^2(\rho')}{k_1^2} - 1\right] E_z(\vec{\rho}\,')G(\vec{\rho};\vec{\rho}\,')\,d\alpha' \qquad (1)$$

In order to come to (1), both permeabilities of the two regions, as shown in Fig. 1, should be equal, otherwise another integral term in (1) should be placed which complicates the problem. Because of the infinite length along z axis, volume integral has been replaced by a surface one. In (1) S is the surface of inhomogeneity, $k_{1,2}$ is the wavenumber of outer and inner region respectively, $G(\vec{\rho};\vec{\rho}\,')$ is the free space cylindrical Green's function [4] while $E_z(\vec{\rho})$ is the unknown field. If $\vec{\rho}\in S$ then $E_z(\vec{\rho})$ represents the stationary field inside the cylinder; if $\vec{\rho}\notin S$ then $E_z(\vec{\rho})$ represents the total field outside the cylinder.

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Expansion of the fields

The incident plane wave illuminates the cylinder normally on the z axis and is impinging with an incident angle of zero degrees. Therefore is has the form [5]

$$E_z^{\text{inc}}(\vec{\rho}) = \sum_{m=0}^{\infty} \varepsilon_m \, j^{-m} J_m(k_1\rho)\cos(m\varphi) \tag{2}$$

where ρ and φ are the polar coordinates with respect to xOy, J_m is the cylindrical Bessel function of the first kind and ε_m is the Neumann factor ($\varepsilon_0 = 1, \varepsilon_n = 2,\ n \geq 1$). The unknown field is expressed in the form

$$E_z(\vec{\rho}) = \sum_{m=0}^{\infty} R_m(\rho)\cos(m\varphi) \tag{3}$$

where $R_m(\rho)$ is the unknown radial function to be evaluated. Then, by substituting (2) and (3) into (1) and applying orthogonality relations for cosines, we arrive at the following integral equation

$$\begin{aligned} R_m(\rho) =& \varepsilon_m \, j^{-m} J_m(k_1\rho) \\ &+ \frac{k_1^2 \pi \varepsilon_m (3-\varepsilon_m)}{4j} \int_0^a \left[\frac{\epsilon_2(\rho')}{\epsilon_1} - 1\right] R_m(\rho') J_m(k_1\rho^<) H_m(k_1\rho^>) \rho' \, d\rho' \end{aligned} \tag{4}$$

where H_m is the Hankel function of the second kind while $\rho^< = \min(\rho,\rho')$ and $\rho^> = \max(\rho,\rho')$.

Expansion in Dini's series

We expand the radial functions in (4) in Dini's series as follows

$$\left[\frac{\epsilon_2(\rho)}{\epsilon_1(\rho)} - 1\right] R_m(\rho) = \sum_{\ell=1}^{\infty} A_{m\ell} J_m\left(\frac{\gamma_{m\ell}}{a}\rho\right), \tag{5}$$

$$R_m(\rho) = \sum_{\ell=1}^{\infty} B_{m\ell} J_m\left(\frac{\gamma_{m\ell}}{a}\rho\right), \text{ and } J_m(k_1\rho) = \sum_{\ell=1}^{\infty} C_{m\ell} J_m\left(\frac{\gamma_{m\ell}}{a}\rho\right). \tag{6}$$

In (5)–(6), $\gamma_{m\ell}$ is the ℓ–th root for every different value of m of the equation [6]

$$\gamma_{m\ell} J_m'(\gamma_{m\ell}) + t_m J_m(\gamma_{m\ell}) = 0 \tag{7}$$

In (7), t_m is an arbitrary parameter (complex number in general) and J_m' is the derivative of Bessel function with respect to its argument.

Solution of the problem

Substituting Dini's series expansions (5)–(6) into (4), putting position vector $\vec{\rho}$ inside the inhomogeneity and using the well known integral of two Bessel functions and a power function [6] as well as Bessel Wronskian relations, one can calculate the unknown coefficients $B_{m\ell}$ of function $R_m(\rho)$ for the internal field by solving the linear systems of equations

$$B_{m\ell} - \sum_{q=1}^{\infty} \left\{ \varepsilon_m \, j^{-m} C_{m\ell} M_{mq} + \frac{(k_1 a)^2}{2} \frac{\varepsilon_m(3-\varepsilon_m)}{N_{m\ell}\left[\gamma_{m\ell}^2 - (k_1 a)^2\right]} G_{m\ell q} \right\} B_{mq} = \varepsilon_m \, j^{-m} C_{m\ell} \tag{8}$$

for $\ell = 1, 2, 3, \ldots$ and for every different value of $m = 0, 1, 2, \ldots$.
In (8), $C_{m\ell}$, $N_{m\ell}$ and M_{mq} are known analytical expressions while $G_{m\ell q}$ is defined by the integral over the inhomogeneity $G_{m\ell q} = \int_0^a \left[k_2^2(\rho)/k_1^2 - 1\right] J_m(\gamma_{mq}\rho/a) J_m(\gamma_{m\ell}\rho/a)\rho\, d\rho$ and is evaluated numerically for every different permittivity profile $\epsilon_2(\rho)$.

The scattered far field

By putting the position vector $\vec{\rho}$ outside the inhomogeneity and carrying out the calculations in (4), we obtain the unknown function $R_m(\rho)$ when $\rho > a$

$$R_m(\rho) = \varepsilon_m\, j^{-m} J_m(k_1\rho) + \frac{k_1^2 \pi \varepsilon_m (3 - \varepsilon_m)}{4j} H_m(k_1\rho) \sum_{\ell=1}^{\infty} C_{m\ell} \sum_{q=1}^{\infty} G_{m\ell q} B_{mq} \tag{9}$$

Using now the asymptotic expansion for the Hankel function in (9), we can calculate the cross section [4] by $\sigma_b = \lim_{\rho\to\infty} \left(2\pi\rho\, |E_z^{\mathrm{sc}}|^2 / |E_z^{\mathrm{inc}}|^2\right)$ and therefore obtain

$$k_1\sigma_b = \frac{(k_1 a)^4}{4}\pi \sum_{m=0}^{\infty} \varepsilon_m\,(3 - \varepsilon_m)\, j^m \cos(m\phi) \sum_{\ell=1}^{\infty} C_{m\ell} \sum_{q=1}^{\infty} G_{m\ell q} B_{mq} \tag{10}$$

Acceleration of convergence

It is apparent from (10) that, in order to calculate $k_1\sigma_b$, the coefficients B_{mq} of the internal field are required. Carrying out asymptotic analysis for B_{mq} reveals that $B_{mq} = O(q^{-3/2})$ for arbitrary values of t_m while, by putting in t_m a special value, it emerges that $B_{mq} = O(q^{-7/2})$. Therefore the series in (10) are optimized and converge faster. The aforementioned special value is obtained from (9) and is [7] $-aR'_m(a)/R_m(a)$ with $R'_m(a)$ being the derivative of (9) with respect to ρ at $\rho = a$.

NUMERICAL RESULTS AND DISCUSSION

In Table 1, the values of back and forward scattering cross section are given for various values of k_1a and for $\epsilon_2(\rho)/\epsilon_1 = 2.45 + 0.5\sin(\pi\rho/a + 0.5)$. We have also compared and verified some results of our method to a high degree of accuracy with the analytical procedure presented in [2].

In Fig. 2, the bistatic scattering cross section is plotted for observation angles from $0°$ to $180°$. This is done for two different permittivity profiles, linear and sinusoidal, and for various values of k_1a. The results are symmetric about $\varphi = 180°$ as it is imposed by the geometry of the scatterer. We also observe the high sensitivity of $k_1\sigma$ to the change of observation angle φ.

In Fig. 3, we depict the behaviour of convergence for the sum versus q in (10) using the optimum and an arbitrary value for t_m. The desired accuracy is achieved with much less terms, when we use the optimum values for t_m in (7) for finding the roots $\gamma_{m\ell}$.

It is noticed that we have verified analytically the known solution, when permittivity profile is constant [4] for arbitrary values of t_m. Finally, this method can be expanded for permittivity profiles which depend on both ρ and φ.

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Table 1: Values of back and forward scattering cross section for $\epsilon_2(\rho)/\epsilon_1 = 2.54 + 0.5\sin(\pi\rho/a + 0.5)$ and comparison with others.

$k_1 a$	$k_1\sigma_\mathrm{b}$ $(\varphi = 180°)$	$k_1\sigma_\mathrm{f}$ $(\varphi = 0°)$	
3.4π	17.585	4.3927	
3.6π	36.495	8.6822	
3.8π	48.287	1.2574	
Comparison: $\lvert E_z^\mathrm{sc}\rvert$ at $\rho/a = 2, \varphi = 0°$ for $\epsilon_2(\rho)/\epsilon_1 = 1 + \beta(\rho/a)^2$			
$k_1 a$	β	This paper	[2]
1.4π	1.2	1.88669509	1.88669443
2.4π	1	2.445371720	2.445359179

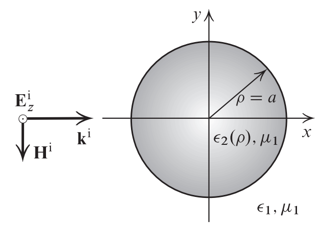

Figure 1: The geometry of the scatterer.

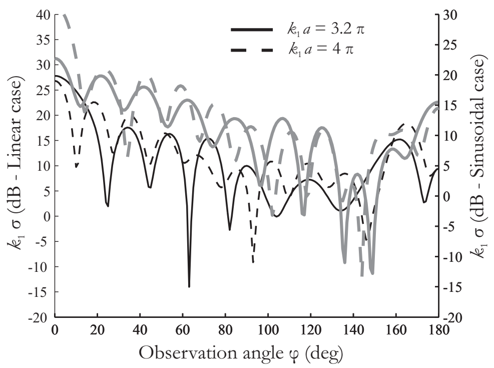

Figure 2: Normalized cross section in dB $[10\log(k_1\sigma)]$. Linear case (black lines): $\epsilon_2(\rho)/\epsilon_1 = 2.54 + \rho/a$. Sinusoidal case (gray lines): $\epsilon_2(\rho)/\epsilon_1 = 2.54 + 0.5\sin(\pi\rho/a + 0.5)$.

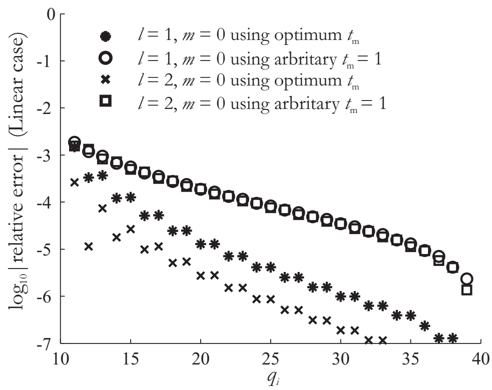

Figure 3: $\log_{10}\lvert\sum_{q=1}^{q_\mathrm{max}} G_{m\ell q}B_{mq} - \sum_{q=1}^{q_i} G_{m\ell q}B_{mq}/\sum_{q=1}^{q_\mathrm{max}} G_{m\ell q}B_{mq}\rvert$ in (10) for $\epsilon_2(\rho)/\epsilon_1 = 2.54 + \rho/a$ and for $k_1 a = 4\pi$.

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